UNIT-3 Numerical analysis

***Numerical Solution of Ordinary Differential equations**: Picard's Method, Taylor's Series, Euler's Method, Modified Euler's Method, Runge-Kutta methods, Milne's and Adam's Bashforth Methods.*

TAYLOR SERIES METHOD:

Consider the initial-value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. then its solution $y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + ...$

Q 1. Employ Taylor series method to obtain the value of y at x=0.2 for the differential equation $\frac{dy}{dx} = 2y + 3e^x$, $y(0)=0$. Compare the numerical solution with the exact solution.
Ans:* Ans: y(0.2)= 0.8110, Exact y(0.2)= 0.8112, Error= 0.8112-0.8110=0.0002** ***[RGPV- June 2004]

Q 2. Employ Taylor series method to obtain the value of y at at $x = 0.1$ and $x=0.2$ for the differential equation $\frac{dy}{dx} = x^2y - 1$, $y(0)=1$. **Ans: y(0.1)=0.9003,y(0.2)= 0.8023** **[Dec.2013 , *June 2017ME]***

Q 3. Employ Taylor series method to obtain the value of y at x=0.1 for the differential equation $\frac{dy}{dx} = x + y^2$, $y(0)=1$.
Ans: y(0.1)= 1.1165 ***[RGPV- June 2002]***

Q 4. Employ Taylor series method to solve the differential equation $\frac{dy}{dx} = 1 - 2xy$, $y(0)=0$.
Ans: $y_5(x) = x - \frac{2x^3}{3} + \frac{4x^5}{15} + ...$ **[Dec. 2001,*RGPV- June 2017(G)]***

PICARD METHOD (Picard's Method of Successive Approximation)

Let us consider the initial-value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

then $y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)dx$, $y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)dx$,……….. $y_{n+1} = y_0 + \int_{x_0}^{x} f(x,y_n)dx$

Q 5. Find the Solution of ODE. Up to 5th approximation using Picard Method $y = 1$, when x = 0 , $y' = y + x$
Ans: $y_5(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$ **[June.12]**

Q 6. Use Picard's Method to obtain the value of y for x=0.2 of the differential equation $\frac{dy}{dx} = x - y$, given that y =1 when x=0 *Ans: y(0.2)=0.83746* ***[RGPV- 2000,2001]***

Q 7. Perform two iterations of Picard's method to find the approximate solution of the initial value problem $\frac{dy}{dx} = x + y^2$, $y(0)=2$. *Ans:* $y_2 = 1 + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}$ ***[RGPV- June 2008 June 2017ME]***

Q 8. Using Picard's process of successive approximation , to obtain a solution up to 5th approximation of the equation $\frac{dy}{dx} = x + y$, $y(0)=1$. ***[RGPV- June 201]2***

Q 9. *Use Picard's Method* to obtain the value of y for x=0.1 of the differential equation $\frac{dy}{dx} = x - y^2$, given that y =1 when x=0 ***[RGPV- Dec 2012]***

Q 10. *Use Picard's Method* to obtain the value of y for x=0.1 of the differential equation $\frac{dy}{dx}=\frac{y-x}{y+x}$, given that y =1 when x=0 *Ans: y(0.1)=1.0906* ***[RGPV- June 2003]***

Q 11. Use Picard's Method to obtain the value of y for x=0.1 of the differential equation $\frac{dy}{dx}=3x+y^2$, given that y =1 when x=0 ***Ans: y(0.1)=1.127*** ***[RGPV- June 2003]***

Q 12. Use **Picard's** method to find the approximate value of *y(0.1)* and *y(0.2)* , given that $y'=x+y^2$, $y(0)$ = 1
Ans: y(0.1)= 1.1165, y(0.2) = 1.2734 **[June 2008, Dec 11]**

Q 13. Using Picard method of successive approximation find y in the interval $0\le x\le 0.5$, given that $\frac{dy}{dx}=y^2-x^2$, $y(0)=0$ (taking h=0.1) **Ans:**

x=	0	0.1	0.2	0.3	0.4	0.5
y=	1	1.11	1.24	1.39	1.56	1.74

EULER'S METHOD : Given the initial-value problem $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$ defined on the interval $x_0 \le x \le x_0+h$, then at $\mathbf{x_1=x_0+h}$, $x_2=x_1+h$…..the **approximate value** of $y(x_0+h)$, denoted by y_1, is given by

$$y_1=y_0+h[f(x_0,y_0)]\ ,\quad y_2=y_1+h[f(x_1,y_1)]\quad \ldots\ldots\quad y_n=y_{n-1}+h[f(x_{n-1},y_{n-1})]$$

Q 14. Find an approximate value y for $x=0.1$, given that $\frac{dy}{dx}=\frac{y-x}{y+x}$ with y(0)=1 . *Ans: y(0.1)=1.0928*

Q 15. Using Euler's method solve the differential equation $\frac{dy}{dx}=x+y$, *y(0)=0, take h=0.2* ***Ans : 0.785984[Dec. 03]***

Q 16. Find *y(2.2)* using Euler's method ,from the equation $\frac{dy}{dx}=-xy^2$, *y(2)=1 take h=0.04,* ***Ans: y(2.2)=0.6907 [June 2011]***

Q 17. Find *y(0.6)* using Euler's method ,from the equation $\frac{dy}{dx}=1-2xy$, *y(0)=0, take h=0.2* , ***Ans: y(0.6)=0.52256 [June 2006]***

Q 18. Using Euler's method solve for y at x=0.1 from $\frac{dy}{dx}=x+y+xy$, *y(0)=1, take h=0.025.* ***[June 2017ME]***

Q 19. Using **Euler's** method find y at $x=1$, for $y'=x^2+y^2$ and y = 1 at x = 0 . **[Dec. 08]** .

Q 20. Find $y(0.04)$ using Euler's method ,from the equation $\frac{dy}{dx}=-y$, *y(0)=1 take h=*0.01, ***[RGPV- May 2019]***

Modified Euler's method
First find y_1 , using Euler's method and then apply modify formula $y_1=y_0+h[f(x_0,y_0)+f(x_1,y_1)]$ where $x_1=x_0+h$ and y_1 is from Euler formula. Similarly Find required approximations.

Alternatively: Find $y_1=y_0+h[x_0+\frac{h}{2},y_0+\frac{h}{2}f(x_0,y_0)]$, $y_2=y_1+h[x_1+\frac{h}{2},y_1+\frac{h}{2}f(x_1,y_1)]$

$$y_{n+1}=y_n+h[x_n+\frac{h}{2},y_n+\frac{h}{2}f(x_n,y_n)]$$

Q 21. Using **Euler's modified method** solve the differential equation $\frac{dy}{dx}=x+y$, *y(0)=1, take h=0.05* ***Ans : 1.052 [Dec. 02, june10]***

Q 22. Solve the following by **Euler's modified method**, the equation $\frac{dy}{dx}=\log_{10}(x+y)$, *y(0)=2* , *at x=1.4* with *h=0.2.*
Ans: y(1.2)=2.5354 , y(1.4)=2.6534 ***[RGPV- Dec. 2004, june05,June 2009]***

Q 23. Find *y(0.2)* by **Euler's modified method**, $\frac{dy}{dx}=\log_{10}(x+y)$, $y(0)=1$ *[**RGPV- May 2019]***

RUNGE-KUTTA METHOD: Given the initial-value problem $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$

Find $k_1=f(x_0,y_0)$, $k_2=f(x_0+\frac{1}{2}h, y_0+\frac{k_1}{2})$, $k_3=f(x_0+\frac{1}{2}h, y_0+\frac{k_2}{2})$, $k_4=f(x_0+h, y_0+k_3)$

$$y_{n+1}=y_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

Solve the equation $\frac{dy}{dx}=x+y$, with initial condition $y(0)=1$ by **Runge-Kutta rule** , from $x=0$ to $x=0.4$ with $h=0.1$.

Ans: y(0)**=1, y(0.1)=1.11034, y(0.2)=1.2428, y(0.3)=1.3997, y(0.4)=1.5836* ***[RGPV- June 2010, 17]

Q 24. Apply Runge-Kutta method of fourth order to solve $10\frac{dy}{dx}=x^2+y^2$, $y(0)=1$ for $x=0.1$

Ans: y(0.1)=1.01013451 ***[RGPV- Dec 2002]***

Q 25. Apply Runge-Kutta method of fourth order to solve $\frac{dy}{dx}=x+y^2$, $y(0)=1$ for $x=0.2$ *in step of 0.1.*

Ans: y(0.2)=1.2736 ***[RGPV- Dec 2004, June ,Dec. 2012]***

Q 26. Apply Runge-Kutta method of fourth order to solve $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ *at* $x=0.2$ *and* $x=0.4$

Ans: y(0.2)=1.196, y(0.4)=1.37527 ***[RGPV- June 2004, 08]***

Q 27. Apply Runge-Kutta method of fourth order to solve $\frac{dy}{dx}=xy$ for x= 1.2 , initially x=1 , y=2 (take h=0.1)

[RGPV- June 03,June 17(G)]

Q 28. Use **Runge-Kutta** method to solve $y'=3x+y^2$ when x=1.1 given that y = 1.2 when x =1 . **[June 05, Dec. 08]**

Milne's Predictor-corrector method: The third-order equations for predictor and corrector are

Let differential equation is $\frac{dy}{dx}=f(x,y), y(x_0)=y_0$

(1) First find four starting values of y values by any previous method(Taylor series, Euler's method, Picard Method , etc.).Find y_0, y_1, y_2, y_3 for x_0 , x_1, x_2, x_3 respectively.

(2) Find y'_0, y'_1, y'_2, y'_3 (*from* $\frac{dy}{dx}=f(x,y)$)

(3) Find y_4 using **Milne's Predictor formula** $y_4=y_0+\frac{4h}{3}[2y_1'-y_2'+2y_3']$ and find $y_4'=f(x_4, y_4)$

(4) Use **Milne's Corrector formula** and find $y_4^{(1)}=y_2+\frac{h}{3}[2y_2'+4y_3'+y_4']$

again $y_4^{(2)}=y_2+\frac{h}{3}[2y_2'+4y_3'+y_4'^{(1)}]$, $y_4^{(3)}=y_2+\frac{h}{3}[2y_2'+4y_3'+y_4'^{(2)}]$……

Continue this process, when two consecutive approximations are equal at desire places

***For This method min. value of n=4**

Q 29. Using **Runge-Kutta** method of order fourth find y for x =0.1 , 0.2, 0.3 given that $\frac{dy}{dx}=xy+y^2$, $y(0)=1$. Continue the solution at x=0.4 using **Milne's method**. **[Dec. 06]**

Q 30. ***Use Milne's Method*** to find the solution of differential equation $\frac{dy}{dx}=x+y$, given that y =0 when x=0 for 0.4< *x<1 with h=0.1* ***[RGPV- Dec 2012]***

Q 31. ***Use Milne's Method*** to find the solution of differential equation $\frac{dy}{dx}=x-y^2$,given that y =0 when x=0 for 0< *x<1*

Ans: y(1)=0.4555 *[* ***RGPV- Dec 2005]***

Q 32. ***Use Milne's Predictor-corrector Method*** to find *y(0.3) from* $\frac{dy}{dx}=x^2+y^2$, given that *y(0)=1.Find the values y(-0.1),y(0.1), y(0.2) from the Taylor's series method.*

Ans: y(0.3)=1.4392 ***[RGPV- Dec 2007]***

Adams-Bashforth-Moulton Method for O.D.E.'s

[1]. Let differential equation is $\frac{dy}{dx}=f(x,y), y(x_0)=y_0$

[2]. First find four starting values of y values by any previous method(Taylor series, Euler's method, Picard Method , etc.).Find y_0, y_1, y_2, y_3 for x_0 , x_1, x_2, x_3 respectively.

[3]. Find y'_0, y'_1, y'_2, y'_3 (*from* $\frac{dy}{dx}=f(x,y)$)

[4]. Tthen apply the predictor formula $y_{n+1}=y_n+\frac{h}{24}[55y'_n-59y'_{n-1}+37y'_{n-2}-9y'_{n-3}]$,

[5]. The corrector formula $y_{n+1}=y_n+\frac{h}{24}[9y'_{n+1}+19y'_n-5y'_{n-1}+y'_{n-2}]$

Q 33. **I**f $\frac{dy}{dx}=x^2(1+y)$ and y(1)=1 , y(1.1)=1.233, y(1.2)1.548, y(1.3)=1.979 the nfind y(1.4) using Adams-Bashforth-Moulton Method. **Ans: Pred. y_1=2.572 , Corr. y_1 =2.572**

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